Title: Running Around in Circles: A Trigonometric Discovery Project

Brief Overview:

This lesson introduces students to trigonometry on the circle. Using the Geometer's Sketchpad, students will be introduced to trigonometric functions and the connection to the geometry on the circle.

Link to Standards:

- **Problem Solving** Students will demonstrate their ability to observe relationships between angles, coordinates, and line segment lengths on the x-y plane.

- **Communication** Students will be able to communicate their conjectures and conclusions both orally and in written form.

- **Reasoning** Students will be able to predict the relationship between trigonometric functions and coordinates on the unit circle.

Grade/Level:

Grades 9-12 (intermediate algebra for introduction; calculus for review).

Duration/Length:

This lesson will take two 45-minute periods or one 70-minute period.

Prerequisite Knowledge:

Students should have working knowledge of the following:

- How to graph points in a coordinate plane
- What a function is
- The Pythagorean Theorem
- Optional: the geometric mean and geometry of similar triangles

Objectives:

Students will :

- learn to use the Geometer's Sketchpad.
- collect data from the software.
- discover connections between geometry and trigonometry.
- visualize the ideas of introductory trigonometry.

Materials/Resources/Printed Materials:

- Macintosh computer and Geometer's Sketchpad
- Student worksheet and instructions

Development/Procedures:

- Arrange students in groups of 2-3 at each Macintosh computer.
- Pass out copies of the student worksheet. This has instructions about how to use the Geometer's Sketchpad. The students may need some help if they do not follow the directions carefully, but they should be self-directed.
- Ask students to record their data and make their conclusions on the worksheet packet.
- Use subsequent class time to discuss more of the algebra or the geometry of trigonometric functions.
- Use the computer portion as a demonstration for a class, if only one computer is available. Students will still use the worksheet to record data and make conclusions.

Evaluation:

Student assessment is based upon the answers that they provide on the worksheet. Further assessment can be based upon their oral or written explanations which summarize what they have learned after spending time with the Sketchpad investigation.

Extension/Follow Up:

- Some Extension units are (or will be) included at the end of the student worksheet. These extensions will allow the students to investigate (1) the tangent, cotangent, secant, and cosecant functions, (2) results when the circle has a radius different from one, and (3) some of the relationships between the six trigonometric functions, based on the geometry of the circle and the Pythagorean theorem.

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A Trigonometric Discovery Project

Running Around in Circles

We are about to embark on a journey of exploration so that you are not trigonometrically challenged. During the trip we will use Geometer's Sketchpad to discover the world of circular reasoning (in a fashion) !!

Created By David W. Stephens and Gail Kaplan

Expanded from "Air on a G String" included in
Exploring Trigonometry with the Geometer's Sketchpad, by David Shaffer
Key Curriculum Press, 1995

Part I: Making the Diagram

Be sure to follow these directions exactly. We all know how picky computers and teachers of mathematics can be.

1. To begin, click on the Geometer's Sketchpad folder and open GSP.

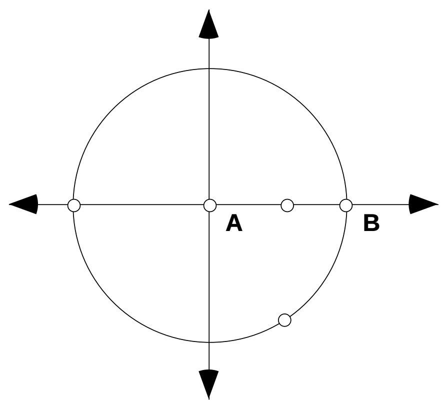
2. Select the **Display** menu and choose **Preferences**. Click on the box **Points**. There should now be an **x** in that box.

3. Our first task is to draw a unit circle, a circle of radius 1.

   a. Select the **Graph** menu and choose **Show Grid**.

   b. Click at the origin. A bold face point will appear. This is point A.
   Depress the shift key and click on the point (1,0). This is point B.
   You should now have two bold face points on your grid.

   c. Select the **Construct** menu and choose **Circle by Center+Point**. A circle with center at the point A and radius 1 will appear.



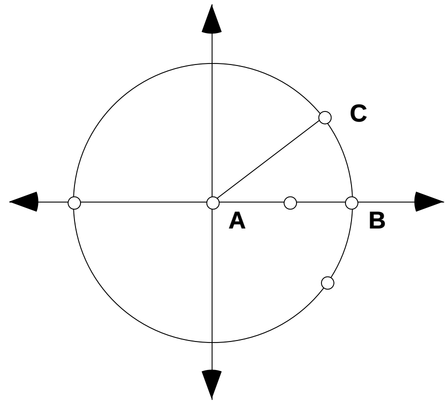
Now that we have a unit circle on the screen, we will draw several line segments which have some unique properties.

4. We will now draw the first line segment.

   a. On the left hand side of the screen there are six boxes. The top box, the arrow box, is now highlighted. Click on the box directly below it, the box with a solid dot.

   b. Click on a point on the circumference of the circle that is also in the first quadrant. Call this point C. Cruisin Charlie lives on point C.

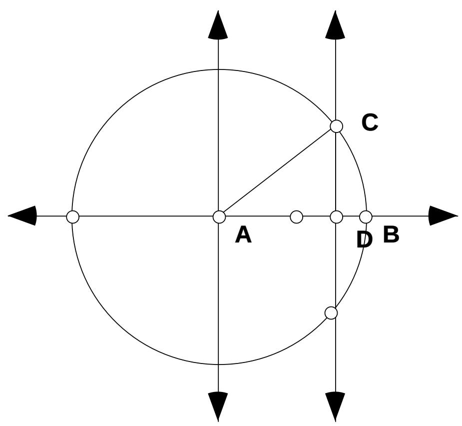
   c. We now want to draw the line from the center of the circle to point C. To accomplish this goal, we need to highlight both points if they are not already highlighted. Click on the box with an arrow on the top of the left hand side of the screen. Click on point A, depress the shift key, and click on point C. Select the **Construct** menu and choose **Segment** to obtain the line segment from point A to point C. What is this line segment called with respect to the circle?



5. We will now draw another line segment.

   a. Click on the x-axis. Depress the shift key and click on the point C. We want to draw a line through C that is perpendicular to the x-axis. Select the **Construct** menu and choose **Perpendicular Line**. You should now see the desired line.

   b. We will now find the point where the perpendicular line from part a intersects the x-axis. Click on the line. Depress the shift key and click on the x-axis. Select the **Construct** menu and choose **Point at Intersection**. You will see the point of intersection highlighted. Call this point D.

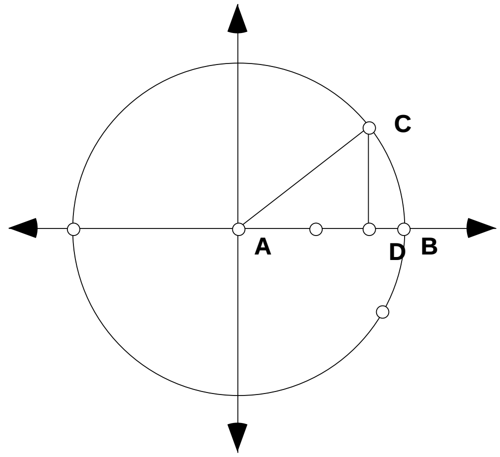


6. Our next goal is to draw the line segment from D to C and from A to D. We do not want to see the entire line from D to C.

   a. Click on the line between C and D. Select the **Display** menu and choose **Hide Line**. The line will magically disappear.

   b. Click on point C. Depress the shift key and click on the point D. Select the **Construct** menu and choose **Segment** to obtain the line segment from point C to point D.

   c. Click on point A. Depress the shift key and click on the point D. Select the **Construct** menu and choose **Segment** to obtain the line segment from point A to point D.

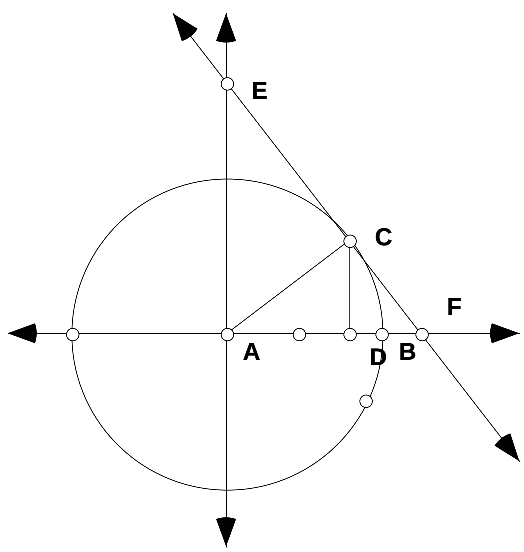


7. The last part of our diagram is to draw the line which is tangent to the circle at the point C and which intersects the x and y axes. Note that the line tangent to a circle at the point C is perpendicular to the radius AC.

   a. Click on the point C, depress the shift key, and click on the line segment AC. Select the **Construct** menu and choose **Perpendicular Line**. You should now see the tangent line.

   b. We will now find the point where the line from part a intersects the y-axis. Click on the line. Depress the shift key and click on the y-axis. Select the **Construct** menu and choose **Point at Intersection**. You will see the point of intersection highlighted. Call this point E.

   c. To mark the point where the line from part a intersects the x-axis click on the line. Depress the shift key and click on the x-axis. Select the **Construct** menu and choose **Point at Intersection**. You will see the point of intersection highlighted. Call this point F.



On your screen you should now have the diagram above including labels.

Part II: Measurements and Observations

During this part of our investigation, we will examine the relationship between the angle BAC, the coordinates of the point C, and the length of the sides of the triangle ADC.

Measurements

1. Follow these directions to fill in the first row of Table I on the next page.

   a. To measure an angle in geometer's Sketchpad, highlight the arrow box on the left hand side of the screen. Now click on point B, depress the shift key, click on A, and click on C. Select the **Measure** menu and choose **Angle**. The result will appear on the screen. Record your measurement in Table I.

   b. Our next job is to find the coordinates of point C. Click on an empty part of the screen. Click on point C. Select the **Measure** menu and choose **Coordinates**. Record your measurement.

   c. To measure the length of line segments CD, AD, and AC, click on the line segment. Select the **Measure** menu and choose **Length**. Record your measurement. Repeat for each segment to complete the first row of Table I.

2. Follow these directions to fill in the second row of Table I.

   a. To obtain the next set of measurements we need to change angle BAC. This is readily done with our software. Highlight the point C and drag it counterclockwise to another location on the circumference of the circle that is also in quadrant I.

   b. Repeat the directions in step 1.

3. Complete in the remaining rows of Table I (at the top of the next page).

   a. To obtain the remaining rows of the table, highlight the point C and drag it counterclockwise to a location on the circumference that is in quadrant II. Repeat the directions in step 1.

   b. Repeat part a until you have made measurements for two locations in each quadrant.

Table I

Quadrant	Angle BAC	(x, y) (coordinates of c)	length of CD	length of AD	length of AC
I					
I					
II					
II					
III					
III					
IV					
IV					

Observations:

Describe the relationship between the x-coordinate and the length of side AD.

Describe the relationship between the y-coordinate and the length of side CD.

As the point C moves counterclockwise along the perimeter of the circle, the x and y coordinates change.

Table II

For each quadrant indicate if the x and y coordinates are positive or negative.

Quadrant	x-coordinate	y-coordinate
I		
II		
III		
IV		

Table III

For each quadrant indicate if the x and y coordinates are increasing or decreasing as the point is dragged around the circle in that quadrant.

Quadrant	x-coordinate	y-coordinate
I		
II		
III		
IV		

Now we are ready to make some observations about the lengths you have been playing with. First, it will be helpful to ANIMATE the sketch of the unit circle and the associated line segments. Cruisin Charlie, the guy who lives on point C will take a stroll along the perimeter of the circle, traveling in a counterclockwise direction.

4. It is time to create an Animation.

   a. Depress the Shift key, click on point C, and click on the circle.

   b. Select the **Edit** menu, choose the **Action Button**, and choose **Animate** (a screen appears first; then click **Animate**). An Animate button will appear in the upper left portion of your screen.

c. To send Cruisin Charlie on his way, double click on the animate button. Be patient. To end the movement, click the mouse.

OPTIONAL: When Cousin Cray Ola comes to visit, Cruisin Charlie enters a colorful world. We will now color the interior of triangle ADC, so that we can see it more clearly.

a. Click on point A, depress the shift key, and click on points D and C. Select the **Construct** menu and choose **Polygon Interior**. The interior will flash in a striped pattern.

b. Select the **Display** menu, choose **Color,** and click on any color.

c. To send Cruisin Charlie and Cousin Cray Ola on their way, double click on the **Animate** button. Charlie and Cray will move around the circle until we click the mouse to stop their journey.

5. The next task is to concentrate on one line segment while observing the animation.

   a. Click on point C and move it along the perimeter of the circle until it is in Quadrant I and as close to point B as possible. Find the line segment CD so you can watch it.

   b. Double click on the **Animate** button and watch how the length of CD changes as Cruisin Charlie moves around the circle.

   c. Write a description of these changes in the following table.

   d. Repeat step 1 through step 3 for each line segment in the table, being sure to always move Cruisin Charlie as close to point B as possible before he begins his journey.

Table IV

Line Segment	Description in Proper Sentences of Changes in Length of Segment
CD	
AD	
AC	
CF	
CE	
AF	
AE	

Some More Measurements

Recall that a function is like a machine. We put an object into the machine and the machine gives us out an object. It's like a candy machine. You put in money and make a selection. The machine gives you out the candy. Most of you are familiar with the function $f(x) = x^2$. If you put 3 into this machine, that means $x = 3$, the function gives out x^2 or $3^2=9$. We want to examine a very special type of mathematical function. The objects put into this machine are angles and the objects that come out of this function are real numbers. We put an angle into the function and the function gives us back a number associated with that angle. The first function we examine is called sin x, where x is an angle. It is important to remember that we must put an angle into the sine function and it will give us a number.

6. We will now evaluate the sine function.
    a. Move point C to a location in quadrant I. On the left hand side of your screen you will see the summary of the last set of measurements.
    b. Click on m<BAC to highlight this expression (please do not choose a 45 degree angle).
    c. Select the **Measure** menu and choose **Calculate**.
    d. Select the **Function** menu and choose **Sin[**.
    e. Select the **Value**. and choose **<BAC**.
    f. Type) and return.

Table V

Quadrant I: Your screen now shows you the value of sin m<BAC. What other value is closely related to the sin m<BAC? Describe this relationship.

Quadrant II: Move the point C to a location in quadrant II. Evaluate the sin<BAC. What other value is closely related to the sin m<BAC? Describe this relationship. Is this the same relationship as in quadrant I?

Quadrant III: Move the point C to a location in quadrant III. Evaluate the sin<BAC. What other value is closely related to the sin m<BAC? Describe this relationship. Is this the same relationship as in quadrant II?

Quadrant IV: Move the point C to a location in quadrant IV. Evaluate the sin<BAC. What other value is closely related to the sin m<BAC? Describe this relationship. Is this the same relationship as in quadrant III?

The next function we will examine is called the cosine of x (abbreviated cos(x)), where x is an angle. It is important to remember that we must put an angle into the cosine function and it will give us a number.

7. We will now evaluate the cosine function.

    a. Move point C to a location in quadrant I. On the left hand side of your screen you will see the summary of the last set of measurements.
    b. Click on m<BAC to highlight this expression.
    c. Select the **Measure** menu and choose **Calculate**.
    d. Select **Function** and choose **Cos[**.
    e. Select **Value**. and choose **<BAC**.
    f. Type) and return.

Table VI

Quadrant I: Your screen now shows you the value of cos m<BAC. What other value is closely related to the cos m<BAC? Describe this relationship.

Quadrant II: Move the point C to a location in quadrant II. Evaluate the cos<BAC. What other value is closely related to the cos m<BAC? Describe this relationship. Is this the same relationship as in quadrant I?

Quadrant III: Move the point C to a location in quadrant III. Evaluate the cos<BAC. What other value is closely related to the cos m<BAC? Describe this relationship. Is this the same relationship as in quadrant II?

Quadrant IV: Move the point C to a location in quadrant IV. Evaluate the cos<BAC. What other value is closely related to the sin m<BAC? Describe this relationship. Is this the same relationship as in quadrant III?

Part III: More Trigonometric Functions

In the first two parts of this project, you discovered the relationship between two new trigonometric functions, sine and cosine, and the lengths of certain line segments. There are actually six trigonometric functions. Now it is time to look at the lengths of four more line segments (EC, CF, AB, and AE) and try to connect their lengths to the values of the trigonometric functions called tangent, cotangent, secant, and cosecant.

Since you know how to use the Geometer's Sketchpad reasonably well, you will not be led through all of the steps for measurement and moving to various points on the circle.

M e a s u r e m e n t s

I. Choose TWO points in each quadrant, measure angle BAC and record measurements for the lengths of EC, CF, AB, and AE in Table V.

Table V

Quadrant	angle BAC	length of BC	length of CF	length of AB	length of AE
I					
I					
II					
II					
III					
III					
IV					
IV					

II. We will now evaluate the other four trigonometric functions.

a) Record the angles you used in Table V in Table VI below.

b) Use the Measure menu (or a calculator) to evaluate the other four trigonometric functions for these angles. NOTE: tangent(angle) is abbreviated tan(angle); cotangent(angle) is abbreviated cot(angle); secant(angle) is abbreviated sec(angle); cosecant(angle) is abbreviated csc(angle).

Table VI

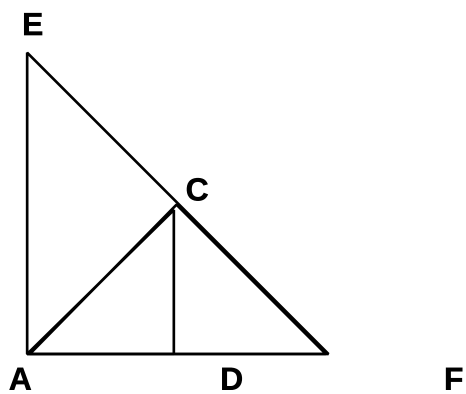
Quadrant	angle BAC	tan(BAC)	cot(BAC)	sec(BAC)	csc(BAC)
I					
I					
II					
II					
III					
III					
IV					
IV					

Observations:

1. Describe any connections you notice between the lengths of the line segments EC, CF, AB, AE and the values of the four new trigonometric functions.

Part IV: Connections Between Geometry and Trigonometry
(Getting the Fundamental Identities)

There are many interrelationships between the six trigonometric functions that you have been investigating in the earlier parts of this project. These relationships can be discovered by combining knowledge of the Pythagorean Theorem with the trigonometric functions that you now know match with specific lengths of line segments. (Once you know the interrelationships, they are very helpful in doing the algebra of trigonometry, as complex trigonometric expressions can be simplified and converted from one form to another. But that is a skill which is beyond where this project is intended to take you.)



Statements (for items 1-5 below, assume that angle ACF is a right angle):

1. a) State the Pythagorean Theorem for triangle ADC with lengths.

   b) Rewrite your answer in terms of trig functions which match with the side lengths.

2. a) State the Pythagorean Theorem for triangle ACF with lengths.

   b) Rewrite your answer in terms of trig functions which match with the side lengths.

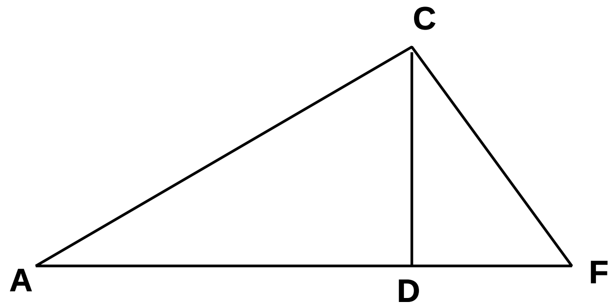
3. a) State the Pythagorean Theorem for triangle ABE with lengths.

   b) Rewrite your answer in terms of trig functions which match with the side lengths.

4. a) State the Pythagorean Theorem for triangle ACE with lengths.

   b) Rewrite your answer in terms of trig functions which match with the side lengths.

Consider this part of the drawing:



5. a) There are three pairs of similar triangles present in this drawing. These similar triangles yield many sets of proportions, some of which have a special feature called "geometric means."

   If you know about geometric means in this situation, then you can ignore the upcoming statements (you can figure them out yourself!).

$$\frac{AD}{CD} = \frac{CD}{DF} \qquad \frac{AF}{AC} = \frac{AC}{AD} \qquad \frac{AF}{CF} = \frac{CF}{DF}$$

   b) Rewrite your answer in terms of trigonometric functions for each of these proportions.

All of the results in part (b) of these five investigations is what is called a TRIGONOMETRIC IDENTITY. There are equal signs between the two sides of the identity. However, these statements are NOT equations. An equation is only true for one or more values of the variable. An identity is ALWAYS true; for any value of the variable, the statement is true.

You will see that the trigonometric identities will be very useful in a variety of applications that use trigonometry. They are used to simplify expressions, to change calculus problems so that they are easier to do, and sometimes to evaluate a problem in a geometry context.

We hope that you have enjoyed the use of the Geometer's Sketchpad to discover some of the fascinating, hidden relationships involved in trigonometry. We hope that the visual nature of this software has helped you to begin to understand some of the complicated and abstract ideas of a branch of algebra and geometry in a way that will help you as you proceed through your study of advanced mathematics.